COMPUTATIONAL INTELLIGENCE WITH QUANTUM GAME'S APPROACH AND ROBUST DECISION-MAKING IN COMMUNICATION INFORMATION UNCERTAINTY

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Abstract

We consider the main ideas and the peculiarities of Quantum Soft Computing tools as a new paradigm of computational intelligence and simulation control processes. Applied Quantum Soft Computing (as a tool and background for robust intelligent control design technology) is discussed. We discuss any important applications (as examples, quantum games and decision-making control processes in quantum uncertainty of information) of this tool in AI-systems.

1. Introduction

We consider the application of the quantum algorithm gate (QAG)-design approach to the classical efficient simulation of quantum games. Games theory is an important branch of economics systems (for example, problems of financial market and global strategy of innovations), social systems, applied mathematics, communications and information transmission, and AI-control systems. It is the theory of decision-making and conflict between different intelligent agents. There are many paradoxes and unsolved problems associated with quantum information and the study of quantum game theory is a useful tool to explore this area [1-6]. In the area of quantum communication, optimal quantum eavesdropping can be treated as a strategic game with the goal of extracting maximal information. For example, the important Benchmarks of quantum computing and information processing as dense coding and teleportation problems can be decided as quantum game models [1]. Using Benchmark's method, different quantum paradigms and methods of AI (on examples from quantum games) are demonstrated. And their applications in problem solution of theoretical informatics (TI) and computer science (Grover's QAG) to design of intelligent robust control systems of essentially non-linear dynamic control objects (intelligent robotics and mechatronics) based on Quantum Soft Computing models can be described. We will study in next step (using described approach and introductory overview) a new problem in applied intelligent control system: design of a *wise* robust control using non-robust particular knowledge bases (KB). This problem is correlated with the solution of *Parrondo* quantum game. It is possible to design a wise robust control from non-robust KB's using quantum computing without entanglement. This approach differs from the methods of quantum games where the entanglement is played the key role.

2. Quantum computing and quantum information processing in AI-systems: quantum game gates approach

We discuss seven (with different physical contents) examples: (*i*) the Prisoner's Dilemma with quantum rules; (*ii*) the Trucker's quantum games; (*iii*) the Quantum Monty Hall Problem; (*iv*) the Parrondo's quantum game; (*v*) the Card game (entanglement-free game); (*vi*) Quantum random walk on a finite lattice; and a collective game (*vii*) Master and pupil - as intelligent models of QAG-computation and quantum communication. Table 1 shows the quantum gates of these quantum games. Some examples of simulation and particular properties of quantum gates from Table 1 below are discussed.

Definitions of quantum game theory and quantum strategies. Any quantum system, which can be manipulated by two parties or more and where the utility of the moves can be reasonably quantified, may be conceived as a quantum game [1].

For example, a two-player quantum game $\Gamma = \{\mathcal{H}, \rho, S_A, S_B, \$_A, \$_B\}$ is completely specified by the underlying. Hilbert space $\mathcal{H}$ of the physical system, the initial state $\rho \in \mathcal{S}(\mathcal{H})$, where $\mathcal{S}(\mathcal{H})$ is the associated state space, the set S_A and S_B of permissible quantum operations of the two players,

Table 1: *Quantum gates of quantum games*

	Game Title	Quantum algorithm gate (QAG)
1	*Prisoner's Dilemma* [1,2]	$\left\|\psi_{fin}\right\rangle = G^{PrDl}\left\|\psi_{in}\right\rangle = G^{PrDl}\left\|\hat{C}\hat{C}\right\rangle = \boxed{\hat{J}^{\dagger}\left(\hat{U}_A \otimes \hat{U}_B \otimes \hat{U}_C\right)}\hat{J}\left\|000\right\rangle$
2	*Trucker's Game* [3]	$\left\|\psi_{fin}\right\rangle = G^{Tr}\left\|00\ldots0\right\rangle = U^{\otimes n}\left\|\psi_{in}\right\rangle = \boxed{U^{\otimes n}QFT\left(p\right)}\left\|00\ldots0\right\rangle$ $= \sum_{j_0=0}^{N-1}\ldots\sum_{j_{N-1}=0}^{N-1} C_{j_0\ldots j_{N-1}}\left\|j_0\ldots j_{N-1}\right\rangle,\quad C_{j_0\ldots j_{N-1}} = \left(\frac{1}{\sqrt{N}}\right)^{N+1}\sum_{k=0}^{N-1}\omega_N^{k\cdot m},$ $m = j_0+\ldots+j_{N-1}+p,\quad QFT\left(p\right)\left\|00\ldots0\right\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\omega_N^{k\cdot m}\left\|kk\ldots k\right\rangle$
3	*Quantum Monty Hall Problem* [4]	$\left\|\psi_{fin}\right\rangle = G^{MHl}\left\|\psi_{in}\right\rangle = \boxed{\left(\hat{S}\cos\lambda + \hat{N}\sin\lambda\right)\hat{O}\left(\hat{I}\otimes\hat{B}\otimes\hat{A}\right)}\left\|\psi_{in}\right\rangle,\ \hat{N} = \neg\hat{S},$ $\hat{O} = \sum_{i,j,k,\ell}\left\|\varepsilon_{ijk}\right\|\left\|njk\right\rangle\left\langle \ell jk\right\| + \sum_{j,\ell}\left\|mjj\right\rangle\left\langle \ell jj\right\|,\ \hat{S} = \sum_{i,j,k,\ell}\left\|\varepsilon_{ij\ell}\right\|\left\|i\ell k\right\rangle\left\langle ijk\right\| + \sum_{i,j}\left\|iij\right\rangle\left\langle iij\right\|,$ $\left\|\varepsilon_{ijk}\right\| = 1, if\ i\neq j\neq k,\ m = \left(j+\ell+1\right)\bmod 3,\ n = \left(i+\ell\right)\bmod 3,\ \lambda\in\left[0,\frac{\pi}{2}\right]$
4	*Parrondo's Paradox* [5]	$\left\|\psi_{fin}\right\rangle = G^{Par}\left\|\psi_{in}\right\rangle = \boxed{\hat{G}^{\otimes n}}\left\|\psi_{in}\right\rangle,\quad where\ \hat{G} = \hat{B}\left(\hat{A}\otimes\hat{A}\otimes\hat{I}\right)$
5	*Card game* [6]	$\left\|\psi_{fin}\right\rangle = \hat{G}^{Card}\left\|000\right\rangle = \left(\boxed{HU_0H}\left\|0\right\rangle\right)\otimes\left(\boxed{HU_1H}\left\|0\right\rangle\right)\otimes\left(\boxed{HU_2H}\left\|0\right\rangle\right);$ $U = U_0\otimes U_1\otimes U_2;\ \left\|r\right\rangle = \left\|r_0\right\rangle\left\|r_1\right\rangle\left\|r_2\right\rangle$
6	*Quantum random walk on a finite lattice L* [7]	$\left\|\psi_{fin}\right\rangle = \hat{G}^{RWalk}\left\|\psi_{in}\right\rangle = \boxed{\left(\left[Sh\right]\otimes\hat{P}_R + \left[Sh^{\dagger}\right]\otimes\hat{P}_L\right)\left(\hat{I}\otimes\hat{U}\right)}\left\|\psi_{in}\right\rangle, \left\|\psi_{in}\right\rangle = \left\|0\right\rangle\otimes\left\|\Phi\right.$ $Sh\left\|n\right\rangle = \left\|n+1\right\rangle, Sh^{\dagger}\left\|n\right\rangle = \left\|n-1\right\rangle, Sh^{L} = I,\ \hat{P}_{R(L)} = \sum_{\alpha=0(M/2)}^{(M/2)-1(M-1)}\left\|\alpha\right\rangle\left\langle\alpha\right\|,$ $\hat{P}_R + \hat{P}_L = \hat{I},$ $\left\|0\right\rangle = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\left\|k\right\rangle, \hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix};\left(\hat{U}\right)_{\alpha\beta} = \frac{1}{\sqrt{M}}\exp\left\{2\pi i\frac{\alpha\beta}{M}\right\};$ $\hat{G}^{RW}\left(\left\|k\right\rangle\otimes\left\|\Phi\right\rangle\right) = \left\|k\right\rangle\otimes\left(e^{-ik}\hat{P}_R + e^{ik}\hat{P}_L\right)\hat{U}\left\|\Phi\right\rangle \equiv \left\|k\right\rangle\otimes\hat{U}_k\left\|\Phi\right\rangle$
7	*Master and Pupil (Dense coding)* [1]	$\left\|\psi_{fin}\right\rangle = \left\|z,\alpha\right\rangle_{AB} = \hat{G}^{MP}\left\|\psi_{in}\right\rangle = \boxed{\mathcal{C}\left(U_{z,\alpha}\otimes I\right)\mathcal{C}^{\dagger}\left\|0\right\rangle_A}\left\|0'\right\rangle_B$ $= \cos\left(\alpha\right)\left\|0'\right\rangle_A\left\|0\right\rangle_B + i\sin\left(\alpha\right)\begin{bmatrix}E_z\left(X\right)\left\|0'\right\rangle_A\left\|1\right\rangle_B + E_z\left(X'\right)\left\|1'\right\rangle_A\left\|0\right\rangle_B\\ +E_z\left(XX'\right)\left\|1'\right\rangle_A\left\|1\right\rangle_B\end{bmatrix},$ $\left\|z\right\rangle = \left\|0\right\rangle + z\left\|1\right\rangle = \left\|0\right\rangle + \left\|z\right\|e^{i\arg(z)}\left\|1\right\rangle = \left\|0'\right\rangle + \frac{1-z}{1+z}\left\|1'\right\rangle,\ E_z\left(\vec{\sigma}\right) = \frac{\left\langle z\right\|\vec{\sigma}\left\|z\right\rangle}{\left\langle z\|z\right\rangle},$ $U_{z,\alpha} = \exp\left\{i\alpha\vec{\sigma}\cdot E_z\left(\vec{\sigma}\right)\right\} = I\cos\alpha + i\vec{\sigma}\cdot E_z\left(\vec{\sigma}\right)\sin\alpha$

and the utility functions $\$_A$ and $\$_B$, which specify the utility for each player. A quantum strategy $s_A\in\mathcal{S}_A$, $s_B\in\mathcal{S}_B$ is a quantum operation, that is, a completely positive trace-preserved (CPTP) map mapping the state space on itself. The quantum game's definition also includes certain implicit rules, such as the order of the implementation of the respective quantum strategies. Rules also exclude certain actions, as the alteration of the payoff during the game.

Remark. Many quantum game's models can be cast into this form. As an example, consider the zero-sum game. A quantum game is called a zero-sum game, if expected payoffs sum up to zero for all pairs of strategies, that is, if $\$_A(s_A, s_B) = -\$_B(s_A, s_B)$ for all $s_A \in \mathcal{S}_A$, $s_B \in \mathcal{S}_B$. Otherwise, it is called a non-zero sum game. It is natural to call two strategies of *A* s_A and s'_A equivalent, if $\$_A(s_A, s_B) = \$_A(s'_A, s_B)$ and $\$_B(s_A, s_B) = \$_B(s'_A, s_B)$ for all possible s_B. That is, if s_A and s'_A yields the same expected payoff for both players for all allowed strategies of **B**. In the same way strategies s_B and s'_B of **B** will be identified. A solution concept provides advice to the players with respect to the action they should take.

According to QAG approach in Table 1 the quantum game can be proceeds as follows.

1.	Starting with a particular initial superposition v, create the entangled state $\hat{J}v$, where $\hat{J}$ is the entanglement operator that communicates with the classical single-player operators
2.	Players select an operation to apply to their part of the superposition, giving $v' = (U_1 \otimes \ldots \otimes U_n)\hat{J}\,v$ where U_k is operator used by player k
3.	Finally undo the initial entanglement, giving $\psi = \hat{J}^{\dagger}v'$. For a given game, i.e., choice for v and $\hat{J}$, the final superposition is a function of the player's choices, i.e., $\psi(U_1, \ldots, U_n)$
4.	Measure the state, giving a specific value for each player's choice. The probability to produce choices s (i.e., a particular assignment, 0 or 1, to each bit) is $\lvert\psi_s\rvert^2$

The following solution concepts are fully analogous to corresponding definitions in standard game theory and will be used.

<u>*Definition*</u>: A quantum strategy s_A is called a dominant strategy of *A* if

$$\$_A(s_A, s'_B) \geq \$_A(s'_A, s'_B)$$

for all $s'_A \in \mathcal{S}_A$, $s'_B \in \mathcal{S}_B$. Analogously we can define a dominant strategy for *B*. A pair (s_A, s_B) is said to be an equilibrium in dominant strategies if s_A and s_B are the player's respective dominant strategies. A combination of strategies (s_A, s_B) is called a Nash equilibrium if

$$\$_A(s_A, s_B) \geq \$_A(s'_A, s_B) \qquad \$_B(s_A, s_B) \geq \$_B(s_A, s'_B)$$

for all $s'_A \in \mathcal{S}_A$, $s'_B \in \mathcal{S}_B$. A pair of strategies (s_A, s_B) is called Pareto optimal, if it is not possible to increase one player's payoff without lessening the payoff of the other player.

Remark. In the quantum game it is only the expectation values of the player's payoffs that are important. For **A** (**B**) we can write as follows: $\langle \$_{A(B)} \rangle = \sum_{i,j=0}^{N-1} P_{ij} \left| \langle \psi_{fin} | ij \rangle \right|^2$, where P_{ij} is the payoff for **A** (**B**) associated with the game outcome $i\,j;\ i, j \in \{0,1\}$.

A solution in dominant strategies is the strongest solution concept for a non-zero sum game. In the Prisoner's Dilemma (see below) defection is the dominant strategy, as it is favorable regardless what strategy the other party picks. Typically, however, the optimal strategy depends on the strategy chosen by the other party. Nash equilibrium (NE) implies that neither player has a motivation to unilaterally after his/her strategy from this equilibrium solution, as this action will lessen his/her payoff. Given that the other player will stick to the strategy corresponding to the equilibrium, the best result is achieved by also playing the equilibrium solution. The concept of NE is of paramount importance to studies of non-zero-sum games with exchange of information between players [8].

Remark. In classical game theory, the player cooperation means that they can completely exchange information with one another, and they take the strategy to the most be hoof of themselves. In this way, cooperators can be seen as one player. It is very important in cooperative game that each party in cooperation must take coordinated strategies. For this purpose, in game theory, NE is an important concept. In a NE, each player obtains his/her payoff, and if he/she tries to change his/her strategy from the NE strategy, his/her payoff will become less. In cooperative quantum game, the situation is similar. *Remark*. It is, however, only an acceptable solution concept if the Nash equilibrium is unique. For games with multiple equilibria the application of a hierarchy of natural refinement concepts may finally eliminate all but one of the NE [8]. Note that NE is not necessarily efficient. In the Prisoner's Dilemma, for example, there is a unique equilibrium, but it is not Pareto optimal, meaning that there is another outcome, which would make both players better off.

Game 1: *The Prisoner's dilemma with quantum rules*. Let us briefly recall the quantum Prisoner's Dilemma presented in [1,2]. Game theory does not explicitly concern itself with how the information is transmitted once a decision is taken. In the Prisoner's Dilemma, the two parties have to communicate with an advocate by talking to her or by writing a short letter on which the decision is indicated. By classical means a two players choice game may be played as follows. An arbiter takes two coins and forwards one coin each to the players. The players then receive their coin with head up and may keep it as it is ("cooperate") or turn it upside down so that tails is up ("defection"). Both players then return the coins to the arbiter who calculates the player's final pay-off corresponding to the combination of strategies he obtains from the players. Here, the coins serve as the physical carrier of information in the game. In quantum version of such a game quantum systems would be used as such carriers of information. For a binary choice two players game an implementation making use of minimal resources involves two qubits as physical carriers.

There are two players have two possible strategies: cooperate $\left(\hat{C} \to |0\rangle\right)$ and defect $\left(\hat{D} \to |1\rangle\right)$. The payoff table for the players is shown in Table 2, with suggestive names for the strategies and payoffs (the case $r = 3, t = 5, s = 0, p = 1$ in [1, 2] is studied).

Table 2: *The general form of the Prisoner's Dilemma payoffs*

Player Strategy	*Bob (**B**)*: $\hat{C}$	*Bob (**B**)*: $\hat{D}$
*Alice (**A**)*: $\hat{C}$	(r, r)	(s, t)
*Alice (**A**)*: $\hat{D}$	(t, s)	(p, p)

The first entry in the parenthesis denotes the payoff of **A** and the second number the payoff of **B**. The entries in this table satisfy conditions: $t > r > p > s$. The meaning of the symbols in the Table 2 is as follows: $C -$ cooperative; $D -$ defect; $r -$ reward; $p -$ punishment; $t -$ temptation; $s -$ sucker's payoff. Classically the dominant strategy for both players is to defect (the *NE*) since no player can improve his/her payoff by unilaterally changing his or her own strategy, even though the *Pareto optimal* is for both players to cooperate. The condition $t > r > p > s$ guarantees that the strategy $\hat{D}$ dominates strategy $\hat{C}$ for both players, and that the unique equilibrium at $\left(\hat{D}, \hat{D}\right)$ is Pareto inferior to $\left(\hat{C}, \hat{C}\right)$. This is the formal description of the dilemma.

<u>Example</u>: *Physical interpretation of formal Prisoner's Dilemma model*. Let us consider the case $r = 3, t = 5, s = 0, p = 1$. The name of the Prisoner's Dilemma arises from the following scenario [2]: two burglars, **A** and **B** are caught by the police and are interrogated in separate cells, without no communication between them. Unfortunately, the police lacks enough admissible evidence to get a jury to convict. The chief inspector now makes the following offer to each prisoner: If one of them confess to the robbery, but the other does not, then the former will get unit reward of 5 units and the latter will get nothing. If both of them confess, then each get 1 unit as a reward. If neither of them confess, then each will get payoff 3. Since confession means a "defect" strategy and no confession means "cooperate" with the other player, the classical strategies of the players are thus denote by "*D*" and "*C*", respectively. Table 2 indicates the payoffs of **A** and **B** according to their strategies. From Table 2, we see that the strategy $\hat{D}$ is the dominant strategy in the game. Since the players are rational and care only about their individual payoffs, both of them will resort to the dominant strategy $\hat{D}$ and get payoffs $p = 1$. In terms of the game theory, $\left(\hat{D}, \hat{D}\right)$ is the dominant strategies equilibrium of the game. However, this dominant strategy equilibrium is inferior to the Pareto optimal $\left(\hat{C}, \hat{C}\right)$, which yields payoffs $r = 3$ to each player's. This is the catch of the Prisoner's Dilemma.

Remark: *Analysis of scenario of the Prisoner's dilemma*. This dilemma is the classical example of non-zero sum game in economics, political science, evolutionary biology, and itself of game theory. A zero sum game is simply a win-lose game. For every turn, as abovementioned the expected payoffs for both players are sum to zero: $\$_A\left(s_A, s_B\right) = -\$_B\left(s_A, s_B\right)$ for all $s_A \in \mathcal{S}_A, s_B \in \mathcal{S}_B$. However, in a non-zero sum game two players no longer appear in strict opposition to each other, but may rather benefit from mutual cooperation. This is what makes these games with non-zero sum interesting. In the Prisoner's Dilemma, two players, **A** and **B**, are pocked up by the police and interrogated in separate cells without a chance to communicate with each other. For the purpose of this game, it makes no difference whether or not **A** or **B** actually committed the crime. The players are told the same thing: If they both choose strategy *D* (defect), they will both get payoff p; if the players both resort to strategy *C* (cooperate), they will both payoff r; if one of the players choose *D* but the other does not, t is payoff for the former and s to the latter (see Table 2). In accordance with the abovementioned Definition a strategy s_A^* can be dominant one of **A** if it satisfies $\$_A\left(s_A^*, s_B\right) \geq \$_A\left(s_A, s_B\right)$ for all $s_A \in \mathcal{S}_A, s_B \in \mathcal{S}_B$. Similarly, we can define a dominant strategy s_B^* for the player **B**. In terms of game theory and abovementioned Definitions, a strategy profile $\left(s_A^*, s_B^*\right)$ can be called a NE if $\$_A\left(s_A^*, s_B^*\right) \geq \$_A\left(s_A, s_B^*\right)$, $\$_B\left(s_A^*, s_B^*\right) \geq \$_B\left(s_A^*, s_B\right)$ for all $s_A \in \mathcal{S}_A$, $s_B \in \mathcal{S}_B$. A NE implies that no players can increase his payoff by unilaterally changing his strategy. A profile $\left(s_A^*, s_B^*\right)$ can be called a Pareto optimal, if it is not possible to increase one player's payoff without lessening the payoff of the other player. A Pareto optimal is a most efficient strategy profile. From the payoff table we can see that *D* is the dominant strategy for both players, i.e., each rational player will choose *D* as his best strategy against his opponent. In addition, mutual defection is a NE. Since the aim of the player is to maximize his own payoff, **A** and **B** will both stick to choosing *D*. But unfortunately, this situation is worse than when they both choose *C*, which happens to be a Pareto optimal. That the NE strategy is not equivalent to the Pareto optimal is the catch of the dilemma in this game.

Example: *Quantum game of two-person Prisoner's Dilemma*. Using the abovementioned gate design method of QA's, the quantum gate of the physical model of the quantum Prisoner's dilemma (originally proposed by *J. Eisert et all* [1]) is shown in Figure 1.

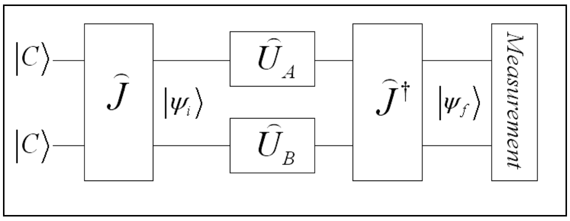

Figure 1: *The gate model for the player quantum Prisoners' Dilemma*

Together with the payoff table for the general Prisoner's Dilemma, the scheme can represent the quantum gate of generalized quantum Prisoner's Dilemma described as the following:

$$\left|\psi_{fin}\right\rangle = G^{PrDl}\left|\hat{C}\hat{C}\right\rangle = \hat{J}^{\dagger}\left(\hat{U}_A \otimes \hat{U}_B\right)\hat{J}\left|\hat{C}\hat{C}\right\rangle \qquad (2.1)$$

In this scheme the game has two qubits, one for each player. The possible outcomes of the classical strategies D and C are assigned to two bases $\left|D\right\rangle$ and $\left|C\right\rangle$ in the Hilbert space of a qubit. Hence, a state of game at each instance is described by a vector in the tensor product space, which is spanned by the classical game basis as following: $\{\left|CC\right\rangle, \left|CD\right\rangle, \left|DC\right\rangle, \left|DD\right\rangle\}$. In the quantum version (see, Figure 1), one starts with the product state $\left|C\right\rangle \otimes \left|C\right\rangle$. One then acts on the state with entangled operator $\hat{J}$ to obtain the initial state of the game as the following: $\left|\psi_{in}\right\rangle = \hat{J}\left|CC\right\rangle = \frac{1}{\sqrt{2}}\left(\left|CC\right\rangle + i\left|DD\right\rangle\right)$, where $\hat{J}$ is a unitary operator, which is known to both players. Strategic moves of **A** and **B** is associated with unitary operators $\hat{U}_A$ and $\hat{U}_B$ respectively, which are chosen from a strategic space $\mathcal{S}$. The players now act with local operators $\hat{U}_A$ and $\hat{U}_B$ on their qubit. Finally, the disentangled operator $\hat{J}^{\dagger}$ is carried out and at the final stage; the state of the game is described by Eq. (2.1). The system is measured in the computational basis, giving rise to one of the four outcomes $\left|CC\right\rangle, \left|CD\right\rangle, \left|DC\right\rangle$ and $\left|DD\right\rangle$, where the first and second entries refer to **A**'s and **B**'s qubits, respectively. The subsequent measurement yields a particular result and the expect payoffs of the players are given by:

$$\begin{cases} \$_A = rP_{CC} + pP_{DD} + tP_{DC} + sP_{CD} \\ \$_B = rP_{CC} + pP_{DD} + sP_{DC} + tP_{CD} \end{cases},$$

where $P_{\sigma\tau} = \left|\left\langle \sigma\tau \middle| \psi_{fin} \right\rangle\right|^2 \left(\sigma, \tau \in \{C, D\}\right)$ is the probability that $\left|\psi_{fin}\right\rangle$ collapsed into basis $\left|\sigma\tau\right\rangle$. If $\hat{U}_A$ and $\hat{U}_B$ are restricted to the classical strategy space $\left(\hat{C} = \hat{I},\, \hat{D} = i\hat{\sigma}_y\right)$, one then recovers the classical game. We can see that expected payoff, for example, **A**'s$\left(\$_A\right)$, not only depends on her choice of strategies $\hat{U}_A$, but also on **B**'s choice $\hat{U}_B$.

Remark. The board of the quantum game is depicted in Figure 1. It can be in fact considered a simple quantum network with sources, reversible one-bit and two-bit gates, and sinks. The complexity is minimal in this implementation as the players' decisions are encoded in dichotomic variables. The quantum game, described by the Eq. (2.1) can be classically efficient simulated using quantum gate approach. Different from the classical game, each player has a qubit and can manipulate it independently (locally) in the quantum version of this game. The quantum formulation proceeds by assigning the possible outcomes of the classical strategies C and D the two vectors of a qubit as $C \to \left|C\right\rangle \to \left|0\right\rangle$ and $D \to \left|D\right\rangle \to \left|1\right\rangle$, respectively. To be specific, the strategy "cooperate" can be associated with the operator, $\hat{C} \equiv \hat{U}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}$, while the strategy "defect" is associated with a spin flip,

$$\hat{D} \equiv \hat{U}(\pi, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In order to guarantee that the ordinary Prisoner's Dilemma is faithfully represented, it is impose the subsidiary commutation conditions

$$\left[\hat{J},\hat{D}\otimes\hat{D}\right]=0,\left[\hat{J},\hat{D}\otimes\hat{C}\right]=0,\left[\hat{J},\hat{C}\otimes\hat{D}\right]=0\cdot \quad (2.2)$$

These conditions, together with the identification $\tilde{J}=\hat{J}^{\dagger}$, imply that, for any pair of strategies taken from the subset $\mathcal{S}_0\equiv\left\{\hat{U}(\theta,0)\middle|\theta\in[0,\pi]\right\}$, the joint probability $P_{\sigma\tau}$ factorize, $P_{\sigma\tau}=P_A^{(\sigma)}P_B^{(\tau)}$, where $p^{(C)}=\cos^2\frac{\theta}{2}$ and $p^{(D)}=1-p^{(1)}=\sin^2\frac{\theta}{2}$. Identifying $p^{(C)}$ with the individual preference to cooperate, we observe that the conditions in Eq. (2.2) in fact ensures that the quantum Prisoner's Dilemma entails a faithful representation of the most general classical Prisoner's Dilemma, where each player uses a biased coin in order to decide whether he or her chooses to cooperate or to defect. Probabilistic strategies of this type are called *mixed* strategies in game theory.

The solution of these conditions (as factoring out Abelian subgroup which yield nothing but a reparametrization of the quantum sector of the strategic space $\mathcal{S}$) is given as the following:

$$\hat{J}=\exp\left\{i\gamma\frac{1}{2}\hat{D}\otimes\hat{D}\right\},\ \gamma\in\left[0,\frac{\pi}{2}\right]. \quad (2.3)$$

A real parameter γ is a measure for the game's entanglement and the gate (2.3) can be considered as the gate, which produces entanglement between the two qubits.

Remark. The game started from the pure state $|C\rangle\otimes|C\rangle$. After passing through the gate $\hat{J}$ as (2.3), the game's initial state is

$$|\psi_{in}\rangle=\hat{J}|CC\rangle=\cos\frac{\gamma}{2}|CC\rangle+i\sin\frac{\gamma}{2}|DD\rangle\cdot$$

Since the entropy (entanglement measure) of $|\psi_{in}\rangle$ is $S=-\sin^2\frac{\gamma}{2}\ln\left(\sin^2\frac{\gamma}{2}\right)-\cos^2\frac{\gamma}{2}\ln\left(\cos^2\frac{\gamma}{2}\right)$ the parameter γ can be reasonably considered as a measure of the game's entanglement [2]. The game's initial state denotes by $|\psi_0\rangle=\hat{J}|CC\rangle$, where $\hat{J}$ is a unitary operator which is known to both players. For fair games, $\hat{J}$ must be symmetric with respect to the interchange of the two players. The strategies are executed on the distributed pair of qubits in the state $|\psi_0\rangle$.

Strategies moves of **A** and **B** are associated with unitary operators $\hat{U}_A$ and $\hat{U}_B$, respectively, which are chosen from a strategic space $\mathcal{S}$. The independence of the players dictates that $\hat{U}_A$ and $\hat{U}_B$ can be operated exclusively on the qubits in **A**'s and **B**'s possession, respectively. The strategic space $\mathcal{S}$ may therefore be identified with some subset of the group of unitary 2×2 matrices. It proves to be sufficient to restrict the strategic space $\mathcal{S}$ to the 2-parameter set of unitary 2×2 matrices.

If one allows quantum strategies of the form:

$$\hat{U}(\theta,\phi)=\begin{pmatrix} e^{i\phi}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\end{pmatrix}, \quad (2.4)$$

with $0\le\theta\le\pi$ and $0\le\phi\le\frac{\pi}{2}$, then there exists a new NE, labeled $\hat{Q}\otimes\hat{Q}$, with the payoff $r=3$ as (3,3). It has the property of being Pareto optimal, therefore the dilemma that exists in the classical game is resolved. If one allows any local operations, then there is no longer a unique NE.

At the beginning of the game the quantum-game qubits $|C\rangle\otimes|C\rangle$ go through an entangling gate J. After the action of both players and another $J^{\dagger}$, the final state $|\psi_{fin}\rangle$ is a superposition. Measurement will make the final state collapse to one of classical outcome and the payoff is returned according to the corresponding entry of the payoff Table 2.

<u>*Example*</u>. This situation was investigated in [1,2]. For a separable game with $\gamma=0$, there exists a pair of quantum strategies $\hat{D}\otimes\hat{D}$, $\hat{D}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ which is the NE and yields payoff (1,1). Indeed, this quantum game behaves "classically", i.e. the NE for the game and the final payoffs for the players are the same as in the classical game. So the separable game does not display any features, which go beyond the classical game. For a maximally entangled quantum game with $\gamma=\frac{\pi}{2}$, there exists a pair of strategies $\hat{Q}\otimes\hat{Q}$, $\hat{Q}=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$, which is a NE and payoff (3,3), having the property to be Pareto optimal. Therefore the dilemma that exists in the classical game is removed. If γ is varied from 0 (no entanglement) to $\frac{\pi}{2}$ (maximally entanglement) then, namely, the quantum game has two thresholds, $\gamma_{th1}=\arcsin\sqrt{\frac{1}{5}}$ and $\gamma_{th2}=\arcsin\sqrt{\frac{2}{5}}$. For $0\le\gamma\le\gamma_{th1}$, the quantum game behaves "classically," i.e., the only NE is $\hat{D}\otimes\hat{D}$ and the payoffs for the players are both 1, which is the same as in classical game. The equilibrium $\hat{D}\otimes\hat{D}$ is no longer a NE because each player can improve his/her payoff by unilaterally from the strategy $\hat{D}$. However, two new NE $\hat{Q}\otimes\hat{D}$ and $\hat{D}\otimes\hat{Q}$ appear. This feature holds for $\gamma_{th1}\le\gamma\le\gamma_{th2}$. In this regime the quantum game does not resolve the dilemma and this domain can be considered as the transitional phase

from classical to quantum. But for $\gamma > \gamma_{th2}$ quantum strategies resolve the dilemma [2]. For $\gamma_{th2} < \gamma \le \frac{\pi}{2}$, a novel NE $\hat{Q}\otimes\hat{Q}$ appears with payoff ($r=3, r=3$). This strategic profile has the property to be Pareto optimal and hence the dilemma disappears, and this domain can be considered as the quantum phase.

Thus, in quantum game NE does not always exist which is totally different from that in classical game. This happens only when initial state is in entangled state. At the same time, when NE exists the payoff function is usually different from that in the classical counterpart except for some special cases.

Remark. As abovementioned, in classical game, NE can be obtained by mixed strategies, where player **A** chooses his two strategies with equal probability. His payoff is zero. Similarly, payoff for the mixed strategy of player **B** is zero too. In quantum game, the players take their strategies by changing the quantum state of the game machine using a unitary operation. A general unitary transformation can be written as [9]

$$\hat{U}(\alpha,\beta)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}, \qquad (2.5)$$

or restrict into the following unitary transformation $\hat{U}(\alpha,\beta)=\sqrt{p}I+i\sqrt{1-p}\sigma_x$, where by changing the parameter p, we can take different strategies. For player **A**, he can choose in principle any operation in the $U(3)$ group. But it is possible set some restrictions, which corresponds to different rules of the game. For instance, in three-dimensional space, the following unitary operation

$$U_3=R_z(\alpha)R_z(\beta)R_z(\theta)=$$
$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

will be one choice. However, this restricted unitary operator is still very complicated because it contains three parameters. It is require the unitary operation to depend on only one parameter.

Remark. Furthermore, the operator can make superposition of all the pure strategies, which is possible in quantum game, but is not possible in classical game. The purpose is to see the effects that brings about by a quantum game machine.

The following operator [9]:

$$U_3=\sqrt{q}I-\sqrt{1-q}\exp\{i\gamma\}M, \quad \gamma=\arccos\left(\frac{1}{2}\sqrt{\frac{1-q}{2q}}\right), \quad M=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is unitary and can produce a superposition of the various strategies. By choosing a different q, **B** chooses different strategies.

Remark. The classical pure strategies correspond to the identity operator $\hat{I}$ and the bit flip operator $\hat{F}\equiv i\hat{\sigma}_x=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Without loss of generality, an entangling operator $\hat{J}(\gamma)$ for an N-player game with two pure classical strategies (an $N\times 2$ game) may be written [5,10] as

$$\hat{J}(\gamma)=\exp\left\{i\frac{\gamma}{2}\hat{\sigma}_x^{\otimes N}\right\}=\hat{I}^{\otimes N}\cos\frac{\gamma}{2}+i\hat{\sigma}_x^{\otimes N}\sin\frac{\gamma}{2},$$

where $\gamma\in\left[0,\frac{\pi}{2}\right], \gamma=\frac{\pi}{2}$ corresponding to maximum entanglement. That is

$$\hat{J}\left(\frac{\pi}{2}\right)|00\ldots0\rangle=\frac{1}{\sqrt{2}}\left(|00\ldots0\rangle+i|11\ldots1\rangle\right).$$

A pure quantum strategy $\hat{U}(\theta,\phi,\alpha)$ is an $SU(2)$ operator and may be written as

$$\hat{U}(\theta,\phi,\alpha)=\begin{pmatrix} e^{i\phi}\cos\frac{\theta}{2} & ie^{i\alpha}\sin\frac{\theta}{2} \\ ie^{-i\alpha}\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix},$$

where $\theta\in[0,\pi]$ and $\phi,\alpha\in[-\pi,\pi]$. A classical mixed strategy can be simulated in the quantum game protocol by an operator in the set $\tilde{U}(\theta)=\hat{U}(\theta,0,0)$. Such a strategy corresponds to playing $\hat{I}$ with probability $\cos^2\frac{\theta}{2}$ and $\hat{F}$ with probability $\sin^2\frac{\theta}{2}$. Where both players use such strategies the game is equivalent to the classical game.

<u>*Example*</u>. In the case of the three players (see Figure 2), without loss of generality [10], we can take

$$\hat{J}=\left(1/\sqrt{2}\right)\left[\hat{I}^{\otimes 3}+i\hat{F}^{\otimes 3}\right], \hat{F}=\hat{\sigma}_x,$$

and the input state $|0\rangle\otimes|0\rangle\otimes|0\rangle=|000\rangle$ becomes $\left(1/\sqrt{2}\right)\left[|000\rangle+i|111\rangle\right]$.

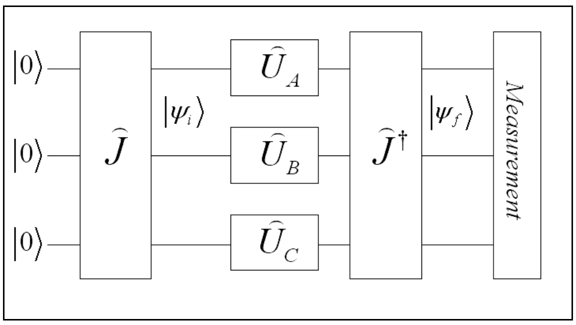

Figure 2: *The set-up gate of three-player quantum games*

For this case the similar analysis of quantum game in [2] is described. In general, players are allowed to apply operator to their qubit(s). We can consider general single-qubit operators, given by [9]

$$\hat{U}(\theta,\phi,\alpha)=\begin{pmatrix} e^{i\phi}\cos\frac{\theta}{2} & e^{i\alpha}\sin\frac{\theta}{2} \\ -e^{-i\alpha}\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix} \quad (2.6)$$

up to an irrelevant overall phase factor. For $n=2$, this reduced to Eq. (2.4). Entangled states allow player 1 to affect the final outcome produced by the action of player 2 and vice versa. Whether an equilibrium exists, and if so whether it is unique and gives the optimum payoffs for the players, depends on the set of allowed operations in quantum gate, the amount and type of entanglement (specified by the choice of $\hat{J}$) and the nature of the payoffs.

We can describe three types of entanglement [9]: (1) Full entanglement; (2) Two-particle entanglement; and (3) Two-particle entanglement with neighbors.

In first case, a conceptually simple approach allows arbitrary entanglement among the player's qubits. As one example, consider fully entangled states. The initial entanglement matrix: $J_n=\left(1/\sqrt{2}\right)\left(I^{\otimes n}+i\sigma_x^{\otimes n}\right)$, where the product in the second term consists of n factors of $\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, the 2×2 Pauli matrix. Allowing general single-bit operators of Eq. (2.5), we can find no pure strategy NE for the players. However, there are a variety of mixed strategy equilibria. As one example, let in Eq. (2.6) as follows:

$$u(0)\equiv U(0,0,0)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, u(1)\equiv U\left(0,\frac{\pi}{2},0\right)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Note $u(0)$ corresponds to the classical "cooperate" operation. A mixed strategy consisting of each player randomly selecting $u(0)$ or $u(1)$, each with probability ½, gives expected payoff $(1+a)/2$ where a is a parameter. This is equilibrium if any one player switches to using a different operator, or different operator's mixture, the expected payoff for that player remains equal to $(1+a)/2$. While this payoff is less than the efficient outcome, it is substantially better than the classical outcome with payoff of 1 since all choose to defect. Full entanglement is difficult to implement as n increases, particularly for qubits communicated over long distance. Thus in second case we consider restricting entanglement to only pairs of qubits. In this case, we suppose each pair of players has a maximally entangled pair, so each player has $n-1$ qubits. The entanglement matrix for a case consisting of $N=\begin{pmatrix}n\\2\end{pmatrix}$ pairs is $J_{pair}(N)=J_2^{\otimes n}$ with the product consisting of N factors of the entanglement operator J_n for the case $n=2$, i.e., full entanglement among two qubits. Two-particle entanglement among all possible pairs of players requires $n(n-1)/2$ entangled pairs. If we consider the players in some arbitrary order and only provide an entangled pair between successive players in that order (with additional pair between the first and last) then this entanglement requires only $2n$ qubits, i.e., the case (3) [9].

Quantum game of three-person Prisoner's Dilemma. In the case of three players from Table 1 (Game1) the entangled three particles have two different types of state: *W*-state and Greenberger-Horne-Zeilinger (*GHZ*) state. When we introduce entanglement to three-person Prisoner's dilemma, it is found in [11] that if the initial state is *W*-state, the players prefer to form a three-person coalition, which can gain the largest payoffs for every player. It is also found that the players prefer to form a two-person coalition and the members of the coalition will gain larger payoffs than the third person if the initial state is GHZ state. From this quantum game it can clearly to see the difference between *W*-state and GHZ-state. The GHZ-state is described as following entangled state:

$$GHZ=\frac{1}{\sqrt{2}}\left(|DDD\rangle+|CCC\rangle\right).$$

And *W*-state is described as following: $W=\frac{1}{\sqrt{3}}\left(|CDD\rangle+|DCD\rangle+|DDC\rangle\right)$. It is two different types of states, which cannot be transformed to each other under local operations and classical transformation in three parties system [12] (see Appendix below). The main differences between these two states are that every two particle in *W*-state have some entanglement while in GHZ-state have not, and three-particle entanglement in GHZ-state is stronger than in *W*-state. When we introduce cooperation into the game, in the classical case it only can form a two-person coalition, that is, it is impossible to form a three-person coalition. When the initial state is *W*-state, the three persons will prefer to form a three-person coalition, which can gain the largest payoffs for each player. When the initial state is GHZ-state, it is unnecessary to form a three-person coalition. But a two-person coalition is inescapable: the players prefer to form a two-person coalition that the members of the coalition will gain larger payoffs than the third person. When the three parties are not entangled, they also prefer to form a two-person coalition, but the third person will gain more payoffs than any members of the coalition.

Classical case. In classical term, the game is described as follows: there are three players in this game, their strategies are $\{C,D\}$. Each 3-triplet stands payoffs of players, respectively, while they use the corresponding strategies. The three persons are symmetric and *D* is the dominant strategy for all of the three players. So the unique equilibrium is DDD, however, there is another strategy CCC would be better for all of the three persons. When two persons are form a coalition it is found that the strategies DCD and DDC are saddle points. Obviously, it is better to be in a two-person coalition than to be an individual player.

Quantum case. For this case there are still three players. There is a source of three bits, with each bit for each player. The strategic space is $\{I, \sigma_x, \sigma_y, \sigma_z\}$. The initial state is produced by the operation $\hat{J}$. So the initial state is $\hat{J}|CCC\rangle$. Since any player of this game uses the strategy σ_x or σ_y, they get the same payoffs, so σ_x and σ_y are equal in this situation. Similarly, strategy I is equal to strategy σ_z. So we only consider two different types of strategy: I and σ_x.

1. When the initial state is *W*-state: $W = \frac{1}{\sqrt{3}}\left(|CDD\rangle + |DCD\rangle + |DDC\rangle\right)$ and there is no cooperation in the game, this game is still a dilemma game. Obviously, I is the dominant strategy for all of the players, so (I, I, I) is an equilibrium. The payoffs are strongly larger than the classical game's equilibrium. But there is another point whose payoffs are larger than the saddle point for all of the players. One may guess that when we introduce cooperation into this game, the payoffs of these players will be increased. It is not completely true. The two-person coalition gains much more payoffs than classical two-person coalition. But there is still another point, which can make the payoffs larger than the saddle point for all of the players. On the other hand, we can find the payoffs of the three persons at the saddle point are the same as the payoffs of the no-cooperation situation, that is, it cannot gain larger payoffs from two-person coalition. But if there exists a three-person coalition, they get more payoffs with everyone agreeing to play σ_x. This means the *W*-state encourages three persons to cooperate together while the classical game only encourages form a two-person coalition.

2. When the initial state is GHZ-state: $GHZ = \frac{1}{\sqrt{2}}\left(|DDD\rangle + |CCC\rangle\right)$ and there is no cooperation among these players. It is found that this game is no longer a dilemma game [11]. The 3-tuples as $(I, I, \sigma_x), (I, \sigma_x, I), (I, \sigma_x, \sigma_x)$ and $(\sigma_x, I, I), (\sigma_x, I, \sigma_x), (\sigma_x, \sigma_x, I)$ are equilibria points. The payoffs of these points are larger than the classical also. When we introduce a two-person coalition to this game, the payoffs matrix has the equilibria points as (σ_x, I, I) and (σ_x, σ_x, I). Also it can be found that the payoff of one player is less than the payoffs of classical two-person coalition. But the coalition gains much more payoffs than the classical one. At the same time, the coalition members gain of two-person more payoffs than non-cooperative case. With this payoffs matrix, we can also say there is no necessary to form a three-person coalition.

We can see that when the initial state is *W*-state, the players prefer to form a three-person coalition, which can gain the largest payoffs for each player. If the initial state is GHZ-state, the players prefer to form a two-person coalition and the members of the coalition will gain larger payoffs than the third person. When the three parties are not entangled, they also prefer to form a two-person coalition, but the third person will gain more payoffs than any members of the coalition.

Remark. From physical point of view GHZ-state has strong three-party entanglement. When initial state is GHZ-state, the strong "cooperation" has been already introduced into this game. So for this situation, the two-person cooperation is a new correlation, which can bring some new results. But the three-person cooperation does not introduce any new correlation to this game, so three-person coalition is not better than two-person coalition in this game. When the initial state is *W*-state, we also know that there is some entanglement in any of the two parties. So when we introduce a two-person coalition to this game, there is no new correlation added and the payoffs of any players in this situation are not better than in no-two-person coalition case. When a three-person coalition is introduced, there is something new and the payoffs of the members in the coalition are increased.

Game 2: *The Trucker's quantum game* [3]. In this game in general case N truckers are planning to go to city B from city A. There are N roads connecting the two cities and all of them are assumed to have the same length. Since none of the trucker's choices, so each of them can only choose his way randomly. The payoff for a certain trucker depends on how many truckers choose the same road as he/she does. The more truckers who choose the same road, the less payoff this certain trucker obtains. If all of them choose the same road, the outcome is the worst because of the possible traffic jamming. The probability of this situation is $P_{worst}^{Cl} = N / N^N$, the superscript *Cl* denotes the classical condition. On the contrary, if they all choose different roads, the probability outcome is the best because there will be no traffic jamming. The probability of this situation is $P_{best}^{Cl} = N! / N^N$. It is obvious that, in order to avoid the worst outcome, the truckers will do their best to avoid choosing the same road. Since they cannot obtain information from each other, the worst outcome will occur with certain probability $P_{worst}^{Cl} = N / N^N$. If we quantize this game, the truckers can definitely avoid the worst outcome by implementing quantum strategies (without knowing what the other truckers choose). The quantum gate of this physical model for described situation is given in Figure 3.

The QAG is described as following:

$$\left|\psi_{fin}\right\rangle = \left(H \otimes H\right)\left|\psi_{in}\right\rangle = \left(H \otimes H\right)\hat{J}\left|00\right\rangle.$$

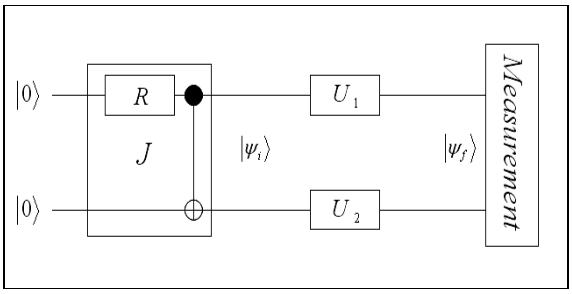

Figure 3: *The set-up gate of the two-trucker game*

We send each player a classical 2-state system in the zero state. The input state is $|00\rangle$. Strategies of truckers are $\hat{U}_A$ and $\hat{U}_B$. The gate $H_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The state after operator $\hat{J}$ is the entangled state $\left|\psi_{in}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$. So, for $\hat{U}_A = \hat{U}_B = H$, where H is Hadamard transformation, the output is $\left|\psi_{fin}\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$. From $\left|\psi_{fin}\right\rangle$ we can see that the two truckers are definitely in different roads. So the probability of the best outcome is 1 and the worst situation of classical case is removed. This is the result that the truckers want.

Remark. A *QFT*-gate is used for to entangle the initial state, which is the state sent to the *N*-truckers. In some cases, the truckers want to guarantee that the pay-off they can at least obtain is better that that when they all choose the same way; the parameter p can be used for the choice of the player payoffs:

if $p = 1$, **then** $\left|\psi_{in}\right\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\omega_N^{k\cdot 1}\left|k\,k\cdots k\right\rangle$.

Hence,

$$C_{j_0\cdots j_{N-1}} = \left(\frac{1}{\sqrt{N}}\right)^{N+1}\sum_{k=0}^{N-1}\omega_N^{k\cdot m},\ m = j_0 + \ldots + j_{N-1} + p\left(=1\right).$$

If $j_0 = j_1 = j_{N-1} = j$ (for which the outcome is the worst) then the coefficient of $\left|j\,j\ldots j\right\rangle$ is

$$C_{jj\ldots j} = \left(\frac{1}{\sqrt{N}}\right)^{N+1}\sum_{k=0}^{N-1}\omega_N^{k\cdot(jN+1)} = 0.$$

So the worst situation will never appear and the payoff of truckers is guaranteed. For this case we see that the worst situation will not occur. The pay-off that the truckers can at least obtain can be definitely better than that of the worst outcome. If the truckers want to increase the probability of the occurrence of the best outcome, the parameter can be set as $p = \frac{N\left(N-1\right)}{2}$ and the probability of the best outcome can be increased. So

$$\left|\psi_{in}\right\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\omega_N^{k\cdot\frac{N(N-1)}{2}}\left|kk\ldots k\right\rangle.$$

While $j_0, j_1, \ldots, j_{N-1}$ are different from each other, there is only one truck in one road and the situation is the best. The probability of this circumstance is $P_{best}^{Q} = N!\cdot\left|C_{01\ldots N-1}\right|^2 = N\cdot\frac{N!}{N^N} = N\cdot P_{best}^{Cl}$, i.e., the quantum probability is N times higher than the classical one. Since the coefficient $C_{j_0\cdots j_{N-1}}$ is either $\sqrt{N/N^N}$ or 0, and the square norm of the coefficient is the probability that the final state (after being measurement) collapses into the corresponding basis, so the probability of the result is $\left|\psi_{fin}\right|^2 = N/N^N$, which is N times higher than in the classical game, or just 0. Whether the probability is 0 or not depends on both the basis and the initial state of the game [3]. Unlike the classical game, in the quantum game by setting different values of parameter *p*, truckers can always meet their various needs, removing the worst outcome to occur, without knowing the choice of other truckers.

Game 3: *Quantum Monty Hall problem*. We discuss the Monty hall problem where the players are permitted to select quantum strategies [4]. It has been suggested that a quantum version of Monty Hall problem may be of interest in the study of quantum strategies of quantum measurement [13].

Classical Monty Hall problem. The player **A** ("Alice") is the banker and can secretly selects one door of three behind which to place a prize (a car). The player **B** ("Bob") picks a door. **A** then opens a different door showing that the prize is not behind it. **B** now has the option of sticking with his current selection or changing to the untouched door. Classically, the optimum strategy for **B** is to alter his choice of door and this, surprisingly, doubles his change of winning the prize from (1/3) to (2/3).

Remark. The classical Monty Hall problem [14] has raised much interest because it is sharply counterintuitive. Also from an information viewpoint it is illustrate the case where an apparent *null operation* does indeed *provide information* about the system.

Classical solution: *Bayes's formula approach*. Game show setting: There are three doors, behind one of which is a prize (car). Monty Hall, the host, asks you to pick a door, any door. You pick door **A** (say). Monty opens door **B** (say) and shows voila there is nothing behind door **B**. Gives you the choice of either sticking with your original choice of door **A**, or switching to door **C**.

Question	*Should you switch*?
Answer	*Yes*

From mathematical statistics point of view we are interested in the following events: C_i is the car that at door $i = 1,2,3$; H_j is the host opens door $j = 1,2,3$. For this case $P(C_i) = \frac{1}{3}$ is a priori probability that the car is at door i. The host's choice of which door to open is made in response to the actual location of the car. The events C_i are the causes that produce the effects H_j. The probabilities of the effects given the causes we call the productive probabilities; there are the conditional probabilities $P(H_j | C_i)$. And $P(H_j)$ is a priori probability that the host opens door j. Bayes's formula is the fundamental equation relating the a posteriori to the productive probabilities:

$$P(H_j)P(C_i|H_j) = P(C_i)P(H_j|C_i),$$

where $P(C_i|H_j)$ are the conditional probabilities that describe the a posteriori probabilities of the causes given the effects.

In our case *a priori* probability that the car is behind door H, $P(H) = \frac{1}{3}$. We can calculate the corresponding probabilities in Bayes's formula as following.

Step	Probability computation algorithm
1	The probability that Monty Hall opens door B if the car were behind A, $P(Monty\ opens\ B|A) = \frac{1}{2}$.
2	The probability that Monty Hall opens door B if the car were behind B, $P(Monty\ opens\ B|B) = 0$.
3	The probability that Monty Hall opens door B if the car were behind C, $P(Monty\ opens\ B|C) = 1$.
4	The probability that Monty Hall opens door B is then $P(Monty\ opens\ B) =$ $P(A)\cdot P(Monty\ opens\ B|A) + P(B)\cdot P(Monty\ opens\ B|B)$ $+P(C)\cdot P(Monty\ opens\ B|C)$ $= \boxed{\frac{1}{6} + 0 + \frac{1}{3} = \frac{1}{2}}$
5	Then, by Bayes's theorem, we obtain the following: $P(A|Monty\ opens\ B) =$ $= \frac{P(A)\cdot P(Monty\ opens\ B|A)}{P(Monty\ opens\ B)} = \frac{1/6}{1/2} = \boxed{\frac{1}{3}}$
6	and $P(C|Monty\ opens\ B)$ $= \frac{P(C)\cdot P(Monty\ opens\ B|C)}{P(Monty\ opens\ B)} = \frac{1/3}{1/2} = \boxed{\frac{2}{3}}$

In other words, the probability that the car is behind door C is higher when Monty opens door B, and you should switch, i.e.,

$$\begin{aligned} &P(you\ win\ if\ \ you\ switch) \\ &= P(H_3 \cap C_2) + P(H_2 \cap C_3) \\ &= P(C_2)P(H_3|C_2) + P(C_3)P(H_2|C_3) \\ &= \frac{1}{3}\times 1 + \frac{1}{3}\times 1 = \boxed{\frac{2}{3}} \end{aligned}$$

Physical interpretation of Monty Hall problem. From physical point of view in the classical case, **A** may put the particle in one of some (such as three) boxes and **B** picks one box. If he finds the particle in the box, he wins (suppose one coin), otherwise, he loses one coin. Obviously, **B** will not accept this proposal, for he has definite 2/3 chance to lose. He may argue that after he chooses one box, **A** should reveal an *empty one from the other two* boxes and then he is provided a chance to choose between sticking with the original choice and switching to the third box. Counter-intuitively, this puts them in a dilemma situation. **B** can win with probability 2/3 by choosing to switch. This case is usually called Monty hall problem. The key point is that when **B** selects a box, he has expected 1/3 chance to win, which will not change anymore. Under the condition that **A** reveals an empty box, **B** may have 2/3 chance to find the particle in the third box. Thus, in classical Monty Hall problem, one player (**B**) can always win with probability 2/3.

Remark. In quantum mechanics an apparent paradox was proposed by *Aharonov* and *Vaidman* [15], in which a single particle can be found with certainty in two (or more) boxes what has come to be called the "*Three-box paradox*." It was introduced a post-selected process which appears to require that each of two disjoint events occurs with certainty. They discuss this in terms of a variable whose three values are the occurrence of a particle within one of three boxes. The process starts with the particle in a state smeared over all three boxes and ends with the particle in another, similarly smeared, state. These states are chosen so that, if the first of the boxes is opened during the process, the particle is certain to be found there, while if the second box is opened, the particle is found there: the particle is certain to be in the first box, and the particle is certain to be in the second box.

The quantum version of Monty Hall problem. A quantum version of the Monty Hall problem may be as follows: there is one quantum particle and three boxes $|0\rangle, |1\rangle, |2\rangle$. **A** selects a superposition of boxes for her initial placement of the particle and **B** then selects a particular box. In this case, it make this a fair game by introducing an additional particle entangled with the original one and allowing **A** to make a quantum measurement on this particle as a part of her strategy [4,5].

Remark. If a suitable measurement is taken after a box is opened it can have the result of changing the state of the original particle in such a manner as to "redistribute" the particle evenly between the other two boxes. In the original game **B** has a 2/3 chance of picking the correct box by altering his choice but with this change **B** has ½ probability of being correct by either staying or switching.

It is possible quantize the original Monty Hall game directly, with no ancillary particles, and allow the banker and/or player to access general quantum strategies [4]: **A**'s and **B**'s choices are represented by qutrits and we suppose that they start in some initial state.

Remark. *Qutrit* is the three-state generalization of the term qubit that refers to a two-state system. A third qutrit is used to represent the box "opened" by **A**.

That is, the state of the system can be expressed as $|\psi\rangle = |oba\rangle$, where a is **A**'s choice of box, b is **B**'s choice of box, and o is the box that has been opened. The initial state of the system shall be designated as $|\psi_{in}\rangle$. The final state of the system is (see Table 1):

$$|\psi_{fin}\rangle = G^{MHI}|\psi_{in}\rangle = \left(\hat{S}\cos\lambda + \hat{N}\sin\lambda\right)\hat{O}\left(\hat{I}\otimes\hat{B}\otimes\hat{A}\right)|\psi_{in}\rangle,$$

where $\hat{A}$ is **A**'s choice operator (or strategy), $\hat{B}$ is **B**'s initial choice operator (or initial strategy), $\hat{O}$ is the opening box operator, $\hat{S}$ is **B**'s switching operator, $\hat{N}$ is **B**'s not switching operator, $\hat{I}$ is the identity operator, and $\lambda \in \left[0, \frac{\pi}{2}\right]$. It is necessary for the initial state to contain a designation for an open box but this should not be taken literally (it does not make sense in the context of the game.) We shall assign the initial state of the open box to be $|0\rangle$.

Physical meaning of operator $\hat{O}$. An operator $\hat{O}$ marks a box (i.e., sets the o qutrits) in such a way that it is anti-correlated with **A**'s and **B**'s choices. Physically it is mean that we should not consider $\hat{O}$ to be the literal action of opening a box and inspecting its contents that would constitute a measurement, but rather it is an operator that marks a box. The coherence of the system is maintained until the final stage of determining the payoff. Mathematically, the open operator $\hat{O}$ is a unitary operator that can be written as following (see Table 1):

$$\hat{O} = \sum_{ijk\ell} |\varepsilon_{ijk}|\, |njk\rangle\langle \ell jk| + \sum_{j\ell} |mjj\rangle\langle \ell jj|,$$

where $|\varepsilon_{ijk}| = 1, if\ i \neq j \neq k$, i.e., are all different and otherwise, $m = (j + \ell + 1)\bmod 3,\ n = (i + \ell)\bmod 3$. The second term applies to states where **A** would have a choice of box to open and is one way of providing a unique algorithm for this choice.

Physical meaning of operator $\hat{S}$. **B**'s switch box operator can be written as (see Table 1):

$$\hat{S} = \sum_{i,j,k,\ell} |\varepsilon_{ij\ell}|\, |i\ell k\rangle\langle ijk| + \sum_{i,j} |iij\rangle\langle iij|,$$

where the second term is not relevant to the mechanics of game but is added to ensure unitarity of the operator. Both $\hat{O}$ and $\hat{S}$ map each possible basis state to unique basis state.

Physical meaning of parameter γ *and operators* $\hat{N}, \hat{A}$ *and* $\hat{B}$. Operator $\hat{N}$ is the identity operator on the three-qutrit state. The $\hat{A} = (a_{ij})$ and $\hat{B} = (b_{ij})$ operators can be selected by the players to operate on their choice of box (that has some initial value to be specified later) and are restricted to members of $SU(3)$. **B** also selects the parameter γ that controls the mixture of staying or switching. In the context of a quantum game it is only the expectation value of the payoff that it is relevant. **B** wins if he picks the correct box, hence $\langle \$_B \rangle = \sum_{ij} \left|\langle ijj | \psi_{fin}\rangle\right|^2$.

A wins if **B** is incorrect, so $\langle \$_A \rangle = 1 - \langle \$_B \rangle$.

Particular cases. Let us consider two cases at initial states: with and without entanglement. In quantum game theory it is convention to have an initial state $|000\rangle$ that is transformed by an entanglement operator $\hat{j}$. Instead we shall simply look at initial states with and without entanglement.

<u>Case 1</u>: *Initial state without entanglement*. Suppose the initial state of **A**'s and **B**'s choice is equal mixture of all possible states with no entanglement. According to this proposal we can calculate as following.

State of game	**Results of computational algorithm**
Initial state $\lvert\psi_{in}\rangle$	$\lvert 0\rangle \otimes \frac{1}{\sqrt{3}}(\lvert 1\rangle + \lvert 2\rangle + \lvert 3\rangle) \otimes \frac{1}{\sqrt{3}}(\lvert 1\rangle + \lvert 2\rangle + \lvert 3\rangle)$
Operator $\hat{O}(\hat{I}\otimes\hat{B}\otimes\hat{A})\lvert\psi\rangle$	$\frac{1}{3}\sum_{ijk}\lvert\varepsilon_{ijk}\rvert(b_{0j}+b_{1j}+b_{2j})(a_{0k}+a_{1k}+a_{2k})\lvert ijk\rangle$ $+\frac{1}{3}\sum_{j}(b_{0j}+b_{1j}+b_{2j})(a_{0j}+a_{1j}+a_{2j})\lvert mjj\rangle$
Operator $\hat{S}\hat{O}(\hat{I}\otimes\hat{B}\otimes\hat{A})\lvert\psi_{in}\rangle$	$\frac{1}{3}\sum_{ijk}\lvert\varepsilon_{ijk}\rvert(b_{0j}+b_{1j}+b_{2j})(a_{0k}+a_{1k}+a_{2k})\lvert ikk\rangle$ $+\frac{1}{3}\sum_{jk}\lvert\varepsilon_{jkm}\rvert(b_{0j}+b_{1j}+b_{2j})(a_{0j}+a_{1j}+a_{2j})\lvert mkj\rangle$
Payoff $\langle \$_B\rangle$	$\frac{1}{9}\cos^2\gamma\sum_{jk}(1-\delta_{jk})\lvert b_{0j}+b_{1j}+b_{2j}\rvert^2\lvert a_{0k}+a_{1k}+a_{2k}\rvert^2$ $+\frac{1}{9}\sin^2\gamma\sum_{j}\lvert b_{0j}+b_{1j}+b_{2j}\rvert^2\lvert a_{0j}+a_{1j}+a_{2j}\rvert^2$

Where $m=(j+1)\bmod 3$. If **A** chooses to apply the identity operator, which is equivalent to her choosing a mixed classical strategy where each of the boxes is chosen with equal probability. **B**'s payoff is $\langle \$_B \rangle = \frac{2}{3}\cos^2\gamma + \frac{1}{3}\sin^2\gamma$, which the same as a classical mixed strategy where B chooses to switch with a probability of $\cos^2\gamma$ (payoff $\frac{2}{3}$) and not to switch with probability $\sin^2\gamma$ (payoff $\frac{1}{3}$).

Remark. The situation is not changed where **A** uses a quantum strategy and **B** is restricted to apply the identity operator (leaving his chose as an equal superposition of the three possible boxes). If both players have access to quantum strategies, **A** can restrict **B** to at most $\langle \$_B \rangle = \frac{2}{3}$ by choosing $\hat{A}=\hat{I}$, while **B** can ensure an average payoff of at least $\frac{2}{3}$ by choosing $\hat{B}=\hat{I}$ and $\gamma=0$ (switch).

Thus this is the NE of the quantum game and it given the same result as the classical game. But the NE is not unique.

<u>Case 2</u>: *Initial state with maximal entanglement*. Let us consider a more interesting case when an initial state with maximal entanglement between **A**'s and **B**'s choices. Similar to the Case 1,we can calculate operator's actions as following.

State of game	Results of computational algorithm
Initial state $\lvert\psi_{in}\rangle$	$\lvert 0\rangle\otimes\frac{1}{\sqrt{3}}\left(\lvert 00\rangle+\lvert 11\rangle+\lvert 22\rangle\right)$
Operator $\hat{O}\left(\hat{I}\otimes\hat{B}\otimes\hat{A}\right)\lvert\psi$	$\frac{1}{3}\sum_{ijk\ell}\lvert\varepsilon_{ijk}\lvert b_{\ell j}a_{\ell k}\lvert ijk\rangle+\frac{1}{3}\sum_{j\ell}b_{\ell j}a_{\ell j}\lvert mjj\rangle$
Operator $\hat{S}\hat{O}\left(\hat{I}\otimes\hat{B}\otimes\hat{A}\right)\lvert\psi_{in}\rangle$	$\frac{1}{3}\sum_{ijk\ell}\lvert\varepsilon_{ijk}\lvert b_{\ell j}a_{\ell k}\lvert ikk\rangle+\frac{1}{3}\sum_{jk\ell}\lvert\varepsilon_{jkm}\lvert b_{\ell j}a_{\ell j}\lvert mkj\rangle$
Payoff $\langle \$_B\rangle$	$\frac{1}{3}\sin^2\gamma\sum_j\left\lvert b_{0j}a_{0j}+b_{1j}a_{1j}+b_{2j}a_{2j}\right\rvert^2 + \frac{1}{3}\cos^2\gamma\sum_{jk}\left(1-\delta_{jk}\right)\left\lvert b_{0j}a_{0k}+b_{1j}a_{1k}+b_{2j}a_{2k}\right\rvert^2$
Limited case to classical strategy $\hat{B}=\hat{I}$	$\frac{1}{3}\sin^2\gamma\left(\lvert a_{00}\rvert^2+\lvert a_{11}\rvert^2+\lvert a_{22}\rvert^2\right) + \frac{1}{3}\cos^2\gamma\left(\lvert a_{01}\rvert^2+\lvert a_{02}\rvert^2+\lvert a_{10}\rvert^2+\lvert a_{12}\rvert^2+\lvert a_{20}\rvert^2+\lvert a_{21}\rvert^2\right)$

Remark. Setting $\hat{B}=\hat{I}$ is equal to the classical strategy of selecting any of the three boxes with equal probability and **B** is limited to a classical mixed strategy.

A can then make the game fair by selecting an operator whose diagonal elements all have an absolute value of $\frac{1}{\sqrt{2}}$ and whose off-diagonal elements all have absolute value $\frac{1}{2}$. One such $SU(3)$ operator and this yields a payoff to both player of $\frac{1}{2}$, whether **B** chooses to switch or not. For a maximally entangled initial state in a symmetric quantum game, every quantum strategy has a counterstrategy since for any $U\in SU(3)$

$$\left(\hat{U}\otimes\hat{I}\right)\frac{1}{\sqrt{3}}\left(\lvert 00\rangle+\lvert 11\rangle+\lvert 22\rangle\right)=\left(\hat{I}\otimes\hat{U}^T\right)\frac{1}{\sqrt{3}}\left(\lvert 00\rangle+\lvert 11\rangle+\lvert 22\rangle\right).$$

It means that for any strategy $\hat{A}$ chosen by **A**, **B** has the counter $A^{\bullet}$ (since the initial choice of players are symmetric):

$$\left(\hat{A}^{\bullet}\otimes\hat{A}\right)\frac{1}{\sqrt{3}}\left(\lvert 00\rangle+\lvert 11\rangle+\lvert 22\rangle\right)=\frac{1}{\sqrt{3}}\left(\lvert 00\rangle+\lvert 11\rangle+\lvert 22\rangle\right),$$

while $\left(\hat{A}^{\bullet}\otimes\hat{A}\right)=\left(\hat{I}\otimes\hat{A}\hat{A}^{\dagger}\right)$. The correlation between **A**'s and **B**'s choices remains so **B** can achieve a unit payoff by not switching boxes.

Remark. Similarly for any strategy $\hat{B}$ chosen by **B**, **A** can ensure a win by countering with $\hat{A}=B^{\bullet}$ if **B** has chosen $\gamma=0$, while a case $\gamma=1$ strategy is defeated by $B^{\bullet}\bullet\hat{M}$ where $\hat{M}$ is amount to a shufting of **B**'s choice, and then switch boxes. As a result there is no NE amongst pure quantum strategies.

Thus, if **B** has access to a quantum strategy and **A** does not he can win all the time. Without entanglement the quantum game confirms the expectations by offering nothing more than a classical mixed strategy. For the Nash equilibrium strategy the player **B** wins two-thirds (i.e., $\frac{2}{3}$) of the time by switching boxes when both participants have access to quantum strategies and maximal entanglement of the initial states produces the same payoffs as a classical game. We will discuss the practical application of this case when one of the player have the access to quantum strategies and another do not have the same access in next step.

3. Role of quantum communication in quantum games

A side is to what extent the quantization procedure blurs the contrast between cooperative and non-cooperative games. In non-cooperative games players are not allowed to communicate, cannot enter binding agreements, and, importantly, cannot use correlated random variables. However, by giving the players and entangled quantum state, one allows them in principle to make use of correlations present in such a state, violating the spirit of a non-cooperative game. Moreover, when comparing quantum and classical versions of a game one should of course not turn a non-cooperative classical game into an explicitly cooperative quantum version. For instance, the solution of the quantum version of the three-person Prisoner's Dilemma given above is valid only if the players enter a binding agreement to accept one of the three players to win in an *a priori* symmetric game. In the quantum games, we can see that in the decision-making step the player has means of communication with each other, i.e., no one has any information about which strategy the other player will adopt. This is the same as in classical game. A fascinating property in quantum game is *entanglement*. Although there is no communication between the two players, the two qubits are entangled, and therefore one player's local action on his qubit will affect the state of the other. Entanglement plays as a contract of the game [1,2, 16].

<u>Example</u>: *Prisoner's Dilemma quantum communication*. Let us use the following property of Pauli matrices:

$$\sigma_z^j \sigma_x^k : |0\rangle \to (-1)^{jk} |k\rangle, |1\rangle \to (-1)^{j\bar{k}} |\bar{k}\rangle;$$
$$\sigma_x : |0\rangle \to |1\rangle, |1\rangle \to |0\rangle; i\sigma_y : |0\rangle \to -|1\rangle, |1\rangle \to |0\rangle$$

A decides to perform strategy $\sigma_x = U_A$, $j = 1 \& k = 0$. **B** performs strategy $i\sigma_y = U_B$, $j = 1 \& k = 1$. Since in dilemma the entangled state created by $\hat{J}$ is $|\psi_{in}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, after **A** and **B** apply their unitary transformations to their qubits, the following phase interaction between qubits takes place: $(U_A \otimes U_A)|\psi_{in}\rangle = -\frac{1}{\sqrt{2}}(|01\rangle + i|10\rangle)$. Finally, due to the interference caused by $\hat{J}^{\dagger}$, we get

$$|\psi_{fin}\rangle = \hat{J}^{\dagger}\left(-\frac{1}{\sqrt{2}}(|01\rangle + i|10\rangle)\right) = -\frac{1}{2} \cdot 2 \cdot |01\rangle + \frac{i}{2} \cdot 0 \cdot |10\rangle = -|01\rangle$$

Therefore, the state $|01\rangle$, in this case, measured by the jailer and from payoff both players are known the changing of qubits. Thus, it can be said that the payoff returns to each prisoners includes the phase information of each qubit and this way it becomes possible to distinguish a phase difference in a quantum game. The communication that is meaningless in a classical game can be meaningful in a quantum game [2, 17]. If there would not be any other information carrier, then one qubit could contain only one-bit information. In quantum game, two-bit information is exchanged between prisoners. This implies that there is another information carrier, which is the phase of each qubit. The phase difference cannot be distinguished by the simple measurement and a certain device is necessary to do this. It is a series of unitary transformations as $\hat{J}^{\dagger}\left(\hat{U}_A \otimes \hat{U}_B\right)\hat{J}$. In these cases devising the series of unitary transformations from Table 1 is equivalent to designing of QAGs and it can be classically efficient simulated using QAG approach [18].

4. Conclusion

We are discussed different models of quantum games and the role of QAGs in simulation of quantum games on classical computers. We discussed new methods of quantum control decision-making process simulation based on application of quantum strategies with applications to AI, applied informatics and computer science. The QAG's design method on some quantum games are illustrated. It is the background for R&D of "*classical-quantum*" games model. New effect of wise control design from non-robust KBs as "*classical-quantum*" game is based on this model representation [19]. The developed analysis and synthesis of QAG's dynamic are also the background for silicon circuit gate design and simulation of robust knowledge base (KB) for intelligent fuzzy controllers.

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Appendix: *Entanglement of W - and GHZ -quantum states under local operations and classical communication (LOCC) with nonzero probability* [12]. Invertible local transformations of a multipartite system are used to define equivalence classes in the set of entangled states. Two states have the same kind of entanglement if both of them can be obtained from the other by means of LOCC with nonzero probability. When applied to pure states of a three-qubit, this approach reveals the existence of two inequivalent kinds of tripartite entanglement, for which the GHZ state and a W state appear as remarkable representatives. Two randomly chosen pure states cannot be converted into each other by means of LOCC, not even with a small probability of success. The GHZ state $|GHZ\rangle = (1/\sqrt{2})[|000\rangle + |111\rangle]$ or else a second state $|W\rangle = (1/\sqrt{3})[|001\rangle + |010\rangle + |100\rangle]$ splits the set of genuinely trifold entangled states into two sets which are unrelated under LOCC. That is, if $|\psi\rangle$ can be converted into the state $|GHZ\rangle$ and $|\phi\rangle$ can be converted into the state $|W\rangle$, then it is not possible to transform, even with only a very small probability of success, $|\psi\rangle$ into $|\phi\rangle$ nor the other way round. The state $|W\rangle$ cannot be obtained from a state $|GHZ\rangle$ by means of LOCC and thus one could expect, in principle, that it has some interesting, characteristic properties. The GHZ state can be regarded as the maximally entangled state of three qubits. However, if one of the three qubits is traced out, the remaining state is completely unentangled. Thus, the entanglement properties of the state $|GHZ\rangle$ are very fragile under particle losses. The entanglement of $|W\rangle$ state is maximally robust under disposal of any one of the three qubits, in the sense that the remaining reduced matrices retain, according to several criteria, the greatest possible amount of entanglement, compared to any other state of three qubits, either pure or mixed.

operators B and C. Its of these operators is necessarily invertible, and in particular $|\psi\rangle = A^{-1} \otimes B^{-1} \otimes C^{-1} |\phi\rangle$. We can use local unitaries in order to take $|\psi\rangle$ into the useful standard product form

$$|\psi_{GHZ}\rangle = \sqrt{K}\left(c_\delta |0\rangle|0\rangle|0\rangle + s_\delta e^{i\varphi} |\varphi_A\rangle|\varphi_B\rangle|\varphi_C\rangle\right),$$
$$K = \left(1 + 2c_\delta s_\delta c_\alpha c_\beta c_\gamma c_\varphi\right)^{-1} \in (1/2, \infty), \qquad \text{(A.1)}$$

$|\varphi_A\rangle = c_\alpha |0\rangle + s_\alpha |1\rangle$ etc. All this states are in the same equivalence as the $|GHZ\rangle$ under SLOCC. Indeed, the invertible local operator

$$\sqrt{K}\begin{pmatrix} c_\delta & s_\delta c_\alpha e^{i\varphi} \\ 0 & s_\delta s_\alpha e^{i\varphi} \end{pmatrix} \otimes \begin{pmatrix} 1 & c_\beta \\ 0 & s_\beta \end{pmatrix} \otimes \begin{pmatrix} 1 & c_\gamma \\ 0 & s_\gamma \end{pmatrix}$$

applied to $|GHZ\rangle$ produces the state $|\psi_{GHZ}\rangle$. The generalized W - state can be represented as following:

$$|\psi_W\rangle = \sqrt{a}\,|001\rangle + \sqrt{b}\,|010\rangle + \sqrt{c}\,|100\rangle + \sqrt{d}\,|000\rangle, \qquad \text{(A.2)}$$

where $a, b, c > 0$ and $d = 1 - (a + b + c) \geq 0$.

The parties can locally obtain the state (A.2) form the state $|W\rangle$ by applying of an invertible local operation of the form

$$\begin{pmatrix} \sqrt{a} & \sqrt{b} \\ 0 & \sqrt{c} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \dfrac{\sqrt{3b}}{\sqrt{a}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We note that states within the GHZ class and the W class depend, respectively, on five and three parameters that cannot be changed by means of local unitaries. The class W is of zero measure compared to the GHZ class.